Coding Theory
and
Digital Data Transmission
Lesson IV

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1 Convolutions

The convolution operation is the main reason that Fourier transforms are useful for communications. The convolution of two functions $v(t)$and $w(t)$ of the same variable is:

$$v \star w(t) \equiv \int_{-\infty}^{\infty} v(\lambda) w(t-\lambda)\, d\lambda. \tag{1}$$

easilyWe will deal in the future with the physical interpretation of the convolution operation. For now we will be content with some mathematical properties of the convolution. The convolution operator is associative and distributive:

$$v \star w \equiv w \star v \tag{2}$$

$$v \star (w \star z) \equiv (v \star w) \star z \tag{3}$$

$$v \star (w + z) \equiv (v \star w) + (v \star z) \tag{4}$$

The **convolution theorem** is:

$$v \star w(t) \Longleftrightarrow V(f)W(f) \tag{5}$$

and by duality

$$v(t)w(t) \Longleftrightarrow V \star W(f). \tag{6}$$

This shows the simple result that the Fourier transform of the convolution of two functions is just the product of the Fourier transforms. The proof of this theorem is:

$$\begin{aligned}
\mathcal{F}[v \star w(t)] &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} v(\lambda) w(t-\lambda) d\lambda \right] e^{-i2\pi f t} dt \\
&= \int_{-\infty}^{\infty} v(\lambda) \left[\int_{-\infty}^{\infty} w(t-\lambda) e^{-i2\pi f t} dt \right] d\lambda \\
&= \int_{-\infty}^{\infty} v(\lambda) \left[W(f) e^{-i2\pi f \lambda} \right] d\lambda \\
&= W(f) \int_{-\infty}^{\infty} v(\lambda) e^{-i2\pi f \lambda} d\lambda \\
&= W(f)V(f).
\end{aligned} \tag{7}$$

2 The Dirac Delta Function and Impulses

The Dirac delta function is not a strict mathematical function but rather can be thought of as a generalized function defined by the relationship

$$\int_{t_1}^{t_2} v(t)\delta(t)\,dt = \begin{matrix} v(0) & v_1 < 0 < v_2 \\ 0 & \text{otherwise} \end{matrix} \tag{8}$$

where $v(t)$is any regular continuous function at $t = 0$. If $v(t) = 1$ it then follows that

$$\int_{-\infty}^{\infty} \delta(t)\,dt = \int_{-\epsilon}^{\varepsilon} \delta(t)\,dt = 1 \tag{9}$$

where ε is arbitrarily small. Thus

$$\delta(t) = 0 \qquad t \neq 0. \tag{10}$$

We can find many limits of functions that can approach the Dirac delta function. Some trivial representation is:

$$\delta(t) = \frac{1}{\varepsilon}\Pi\left(\frac{t}{\varepsilon}\right) \qquad \lim \varepsilon \to 0 \tag{11}$$

or

$$\delta(t) = \frac{1}{\varepsilon}\text{sinc}\left(\frac{t}{\varepsilon}\right) \qquad \lim \varepsilon \to 0. \tag{12}$$

2.1 Impulses in Frequency

Impulses in frequency represent phasors. This can be easily seen by taking the inverse Fourier transform of a function in frequency that consists of an number of impulses each at a harmonic of the fundamental frequency f_0

$$V(f) = \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_0). \tag{13}$$

When we take the inverse Fourier transform of this function we get

$$v(t) = \mathcal{F}^{-1}[V(f)] = \sum_{n=-\infty}^{n=\infty} c_n e^{i2\pi n f_0 t} \tag{14}$$

which is simply the Fourier series. Thus the spectrum of a periodic function found with the Fourier transform gives the same discrete result that would have been found by using the Fourier series.

In particular if we take the constant function $v(t) = A$, we can easily see that

$$V(f) = A\delta(f). \tag{15}$$

2.2 Impulses in Time

The Dirac delta function is is a nonphysical function which is a mathematical abstraction. It has infinite energy and physical quantities will always have the function integrated over time or frequency. Nevertheless it is very useful.

By duality we can show that the spectrum of the Fourier transform of an impulse at time $t = 0$ having an amplitude of A is simply the constant A

$$A\delta(t) \Longleftrightarrow A. \tag{16}$$

The same impulse at time $t_0$gives us a modulated transform

$$A\delta(t - t_0) \Longleftrightarrow Ae^{-i2\pi f t_0}. \tag{17}$$

It is interesting to take the inverse transform of the right side of eq. 17 this will give us the interesting relationship

$$\delta(t-t_0) = \mathcal{F}^{-1}[e^{-i2\pi f t_0}] = \int_{-\infty}^{\infty} e^{-i2\pi f t_0} e^{i2\pi f t} df = \int_{-\infty}^{\infty} e^{i2\pi f (t-t_0)} df. \tag{18}$$

With this relationship we can prove that

$$v(t) = \mathcal{F}^{-1}(\mathcal{F}(v(t)) \tag{19}$$

ignoring mathematical concerns such as convergence and changes to the order of intergration variables.

Consider the unit function $u(t)$, defined as zero for $t < 0$ and one for $t > 0$

$$u(t) \equiv \begin{array}{ll} 1 & t > 0 \\ 0 & t < 0 \end{array}. \tag{20}$$

We can easily see by the definition, eq 8, that

$$u(t) = \int_{-\infty}^{t} \delta(s)\, ds, \tag{21}$$

and that at least formally

$$\delta(t) = \frac{du(t)}{dt} \tag{22}$$

3 Problems

Prove equations 2, 4 and 19.