Coding Theory
and
Digital Data Transmission
Lesson V

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1 Impulse Response and Frequency Response

1.1 Linear time-Invariant Systems

A **system** is a physical process that relates the **input** (source or excitation) signal to the **output** (response) signal. Let $x(t)$be the input signal and $y(t)$be the output signal. The system can be viewed as a mathematical mapping of $x(t)$into $y(t)$,

$$y(t) = \mathcal{S}[x(t)]. \tag{1}$$

where $\mathcal{S}$ is the operator that produces the output $y(t)$ given the input $x(t)$, as illustrated in fig. 1.

If this systems satisfies the following two conditions, then the system is called a **linear** system:

$$\mathcal{S}[x_1(t) + x_2(t)] = \mathcal{S}[x_1(t)] + \mathcal{S}[x_2(t)] = y_1(t) + y_2(t) \implies \text{additivity property} \tag{2}$$

and

$$\mathcal{S}[ax(t)] = a\mathcal{S}[x(t)] = ay(t) \implies \text{homogeneity property.} \tag{3}$$

If the system satisfies the following condition, then the system is called a **time-invariant** system:

$$\mathcal{S}[x(t - t_0)] = y(t - t_0) \tag{4}$$

where $t_0$is any real constant. Equation 4 means that a time delayed input will give a timed delayed output.

If the systems is both linear (eqs. 2 and 3) and time-invariant (eq. 4), then the system is called a **linear time-invariant** (LTI) system.

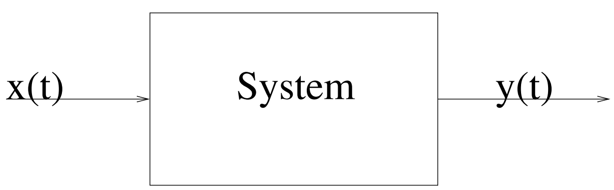

Figure 1: A System mapping $x(t)$ to $y(t)$

1.1.1 Problems

1. For each of the following systems, determine whether the system is linear.

   (a) $\mathcal{S}[x(t)] = x(t)\cos\omega_c t$

   (b) $\mathcal{S}[x(t)] = (A + x(t))\cos\omega_c t$

2. Consider the system with input $x(t)$and output $y(t)$given by

$$y(t) = x(t)\delta_T(t) = x(t)\sum_{n=-\infty}^{n=\infty}\delta(t - nT) \tag{5}$$

   (a) Is this system linear?

   (b) Is the system time-invariant?

1.2 Impulse Response

The **impulse response** $h(t)$ of an LTI system is defined to be the response of the system when the input is $\delta(t)$, that is,

$$h(t) = \mathcal{S}[\delta(t)]. \tag{6}$$

The function $h(t)$ is arbitrary, and need not be zero for $t < 0$. If

$$h(t) = 0 \quad \text{for } t < 0 \tag{7}$$

then the system is called **causal**.

1.3 Response to an Arbitrary Input

The response $y(t)$ of an LTI system to an arbitrary input $x(t)$ can be expressed as the convolution of $x(t)$ and the impulse response $h(t)$ of the system,

$$y(t) = x(t) \star h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau, \tag{8}$$

because of the commutative nature of the convolution we also have

$$y(t) = h(t) \star x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau. \tag{9}$$

1.3.1 Problems

1. Prove equation 8.

2. Show that for a TLI system

$$\frac{dy(t)}{dt} = \mathcal{S}\left[\frac{dx(t)}{dt}\right]$$

3. Prove that if $a(t)$ is a TLI system response to the unit step function $u(t)$(see lecture IV)

$$a(t) = \mathcal{S}[u(t)] \tag{10}$$

   then

$$h(t) = \frac{da(t)}{dt}. \tag{11}$$

1.4 Frequency Response

Applying the time convolution theorem of the Fourier transform to eq. 8, we obtain

$$Y(f) = X(f)H(f) \tag{12}$$

where $Y(f) = \mathcal{F}(y(t))$, $X(f) = \mathcal{F}(x(t))$ and $H(f) = \mathcal{F}(h(t))$. Thus the action of the system on the input signal in frequency space is just a multiplication by, $H(f)$, the Fourier transform of the impulse response of the system $h(t)$. $H(f)$ is called the **frequency response** (or the **transfer function**) of the system. The output signal can be gotten by using the inverse Fourier transform

$$y(t) = \int_{-\infty}^{\infty} X(f)H(f)e^{i2\pi ft}df \tag{13}$$

1.4.1 Problems

1. Show that for a sinusoidal input function $x(t) = Ae^{i2\pi f_c t}$

$$y(t) = H(f_c)x(t).$$

   Hint: Use the eqn 9 and the Fourier integral transform definition. This result shows us that eigenfunctions of the $\mathcal{S}$ operator is just $e^{i2\pi f_c t}$ having eigenvalues $H(f_c)$.

2 Filter Characteristics of LTI Systems

The frequency response of a LTI system, $H(f)$ is in general a complex quantity that can be written as

$$H(f) = |H(f)|e^{i\theta_h(f)}. \tag{14}$$

Since $h(t)$ is a real (not complex) function

$$|H(-f)| = |H(f)| \quad \theta_h(-f) = -\theta_h(f) \tag{15}$$

Similar relationships are true for $X(f)$ and $Y(f)$. Thus, we have

$$|Y(f)| = |X(f)||H(f)| \tag{16}$$

$$\theta_y(f) = \theta_x(f) + \theta_h(f). \tag{17}$$

We see that amplitude of the output frequency function is just the amplitude of the input frequency function multiplied by the amplitude of the transfer function. An LTI system then acts as a filter on the input signal enhancing some frequencies and and diminishing other frequencies. The phase of the output frequency function is just the sum of the phase of the input frequency function and the phase of the transfer frequency function.

3 Transmission of Signal Through LTI Systems

For a transmission to be distortion-less we require that the output signal preserve the same shape as the input signal. If $x(t)$ is the input signal than the output is:

$$y(t) = Kx(t - t_d). \tag{18}$$

Taking the Fourier transform of both sides

$$Y(f) = Ke^{-i2\pi ft_d}X(f). \tag{19}$$

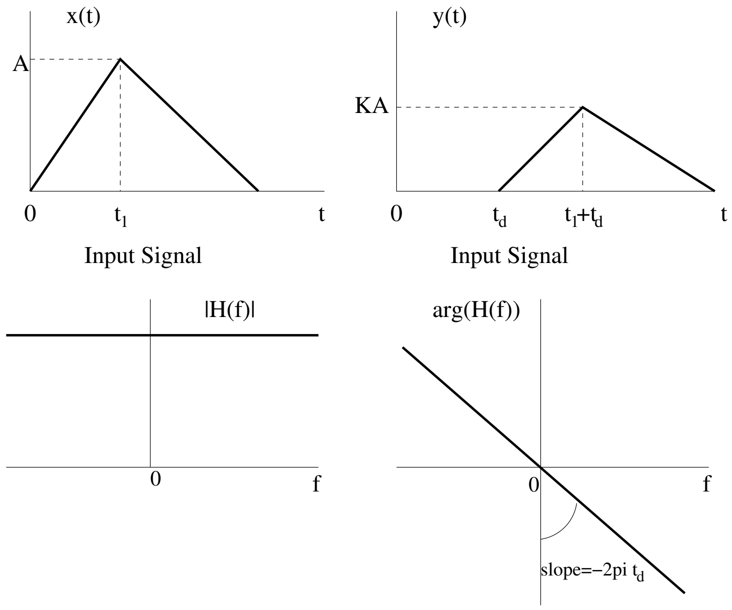

Figure 2: Distortion-less transmission

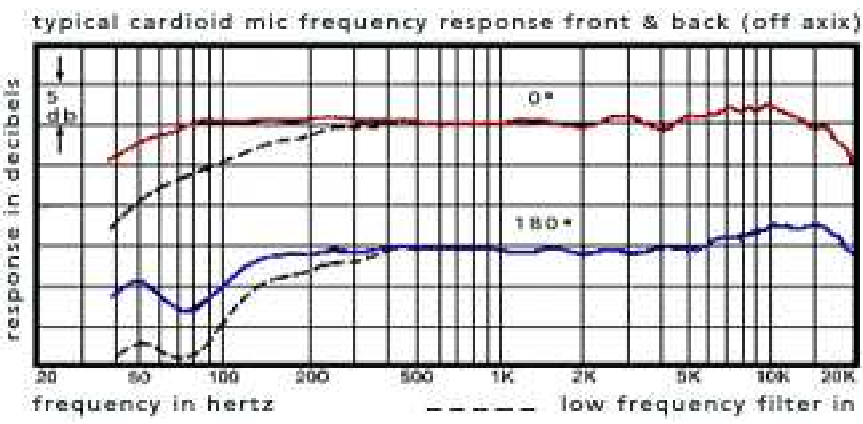

Figure 3: Typical frequency response of a microphone

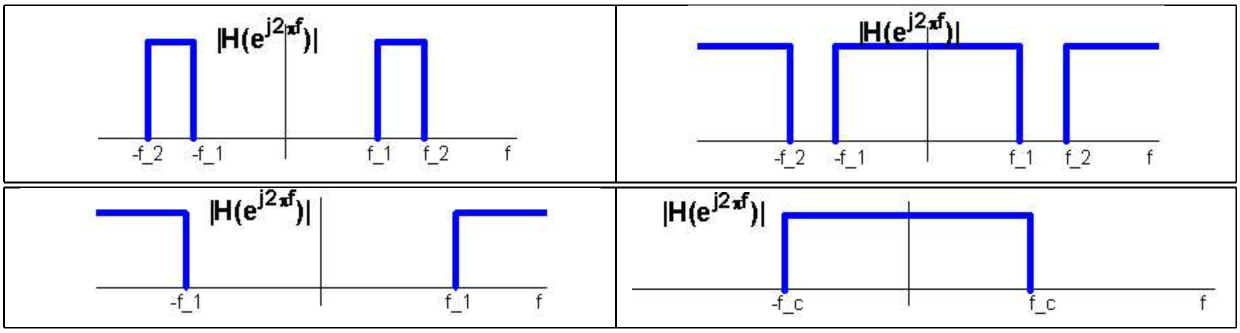

Table 1: Bandpass, Bandstop, High-pass and Low-pass filter

From eq. 19, we must have a transfer function of the form,

$$H(f) = Ke^{-i2\pi f t_d}. \tag{20}$$

Thus the amplitude of $H(f)$ is constant over the entire frequency range of $X(f)$ of interest and the phase of $X(f)$ is linear in frequency. This behavior is shown in fig. 2. The fig. 3 shows the typical frequency response of a microphone.

When the amplitude spectrum $|H(f)|$ of the system is not constant within the frequency of interest, different frequencies are transmitted with different gain or attenuation. This is called **amplitude distortion**.

When the phase spectrum is not linear with frequency, the output signal has a different waveform from the input signal, because of different delays for differing frequencies components of the input signal. This is called **phase distortion**.

4 Filters

By definition an ideal filter has the characteristics of distortion-less transmission for some frequency bands and zero response for at all other frequencies. The **ideal bandpass filter** (BPF) is defined by:

$$H_{BPF}(f) = \begin{cases} e^{-i2\pi f t_d} & \text{for } f_{c_1} \leq |f| \leq f_{c_2} \\ 0 & \text{otherwise} \end{cases}.$$

The ideal bandpass filter passes all input signal components between f_{c_1} and f_{c_2} and rejects all frequency components not within this band of frequencies. The parameters f_{c_1} and $f_{c_2}$are the lower and upper cutoff frequencies. Filters that disallow a particular range of frequencies is called a bandstop filter. Filters that allow frequencies greater than a cutoff is called a high-pass filter. Filters that allow only frequencies less than a cutoff are called a low-pass filter. These four filters are shown in table 1.

Practical filters do not have sharp cutoffs like the ideal filter but rather have smooth frequency response, see fig. 4 The frequency response of a filter is usually defined as the point where the signal value of the filter drops by 3db of the maximum value. At this point the amplitude response drops to approximately $H(0)/\sqrt{2}$. Note: the unit **db** (decibels) is the $-10\log_{10}\frac{P_1}{P_2}$ where P_1 and P_2 are the output and input powers respectively. Since $P \sim A^2$, where the A is the amplitude of the signal the db drop when the response drops $1/\sqrt{2}$ in amplitude is -10log$_{10}\frac{1}{2} = 3.0103$..., this point is usually referred to by engineers as the 3-db bandwidth of the filter.

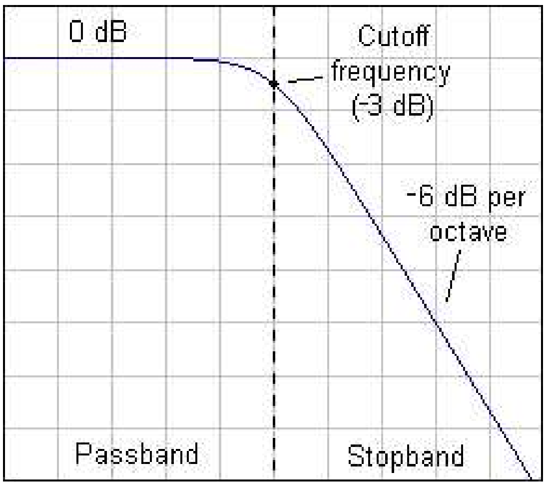

Figure 4: Butterworth Filter