Coding Theory
and
Digital Data Transmission
Lesson VII

Joel Isaacson
Bar-Ilan Computer Science

15th May 2005

As everyone these days knows digital techniques are taking over the older analog techniques. Normally the world that we encounter is analog. Frequently we have to convert analog to digital signals (ADC) and then have to convert the digital signals back to analog signals (DAC) to restore the original analog signal.

1 Pulse Code Modulation (PCM)

PCM consists of three processes, **sampling**, **quantizing** and **encoding** fig. 1.

Sampling is the process of translating the continuous function0.00n in time to a discrete function in time. As we will see there are criteria for when sampling can be done without lose of information.

Quantizing is the process of translating continuous amplitude signals into a discrete number of allowable values. This process usually causes lose of information.

Encoding is the process of preparing the signal in a form that can be efficiently transmitted.

2 Sampling

A **band-limited** signal is a signal, $m(t)$, that has no Fourier components above a certain frequency $\omega_M = 2\pi f_M$:

$$m(t) \Leftrightarrow M(\omega) = 0 \text{ for } |\omega| > \omega_m. \tag{1}$$

Nyquist's theorem states that a band-limited function can be can be recovered completely by sampling at a greater rate (smaller time periods) than the Nyquist sampling rate $T_N = \frac{1}{2f_M}$. Thus if the sampling rate T_S is less than T_N, we lose no information by this sampling. Nyquist's theorem also tells us how the original function, $m(t)$, is related to the sampled values of the function, $m(nT_S)$

$$m(t) = \sum_{n=-\infty}^{n=\infty} m(nT_s) \frac{\sin(2\pi f_M(t - nT_S))}{2\pi f_M(t - nT_S)} = \sum_{n=-\infty}^{n=\infty} m(nT_s) sinc(2f_M(t - nT_S)) \tag{2}$$

Figure 1: Pulse Code Modulation

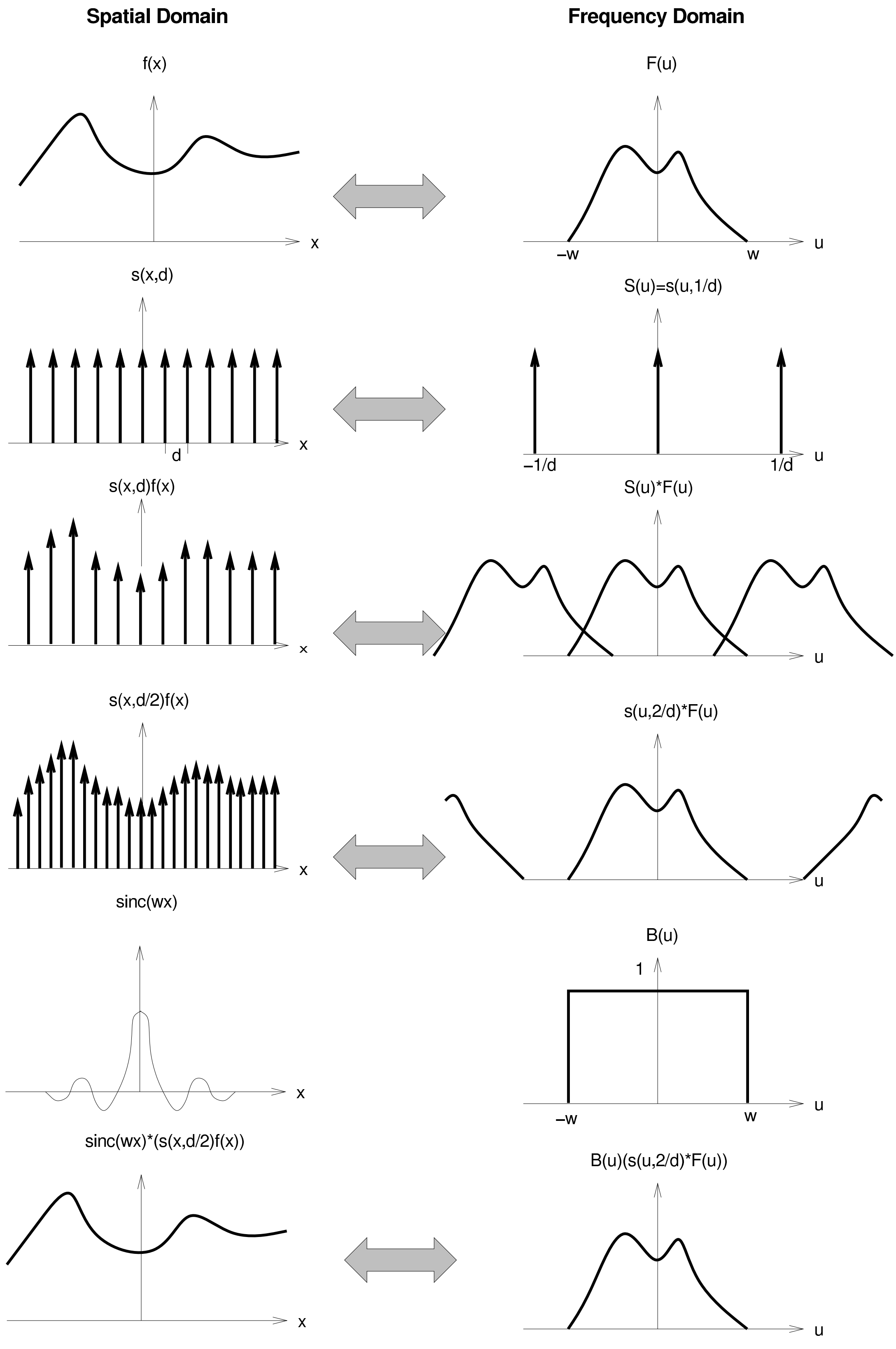

Figure 2: Proof of Nyquist's Theorem

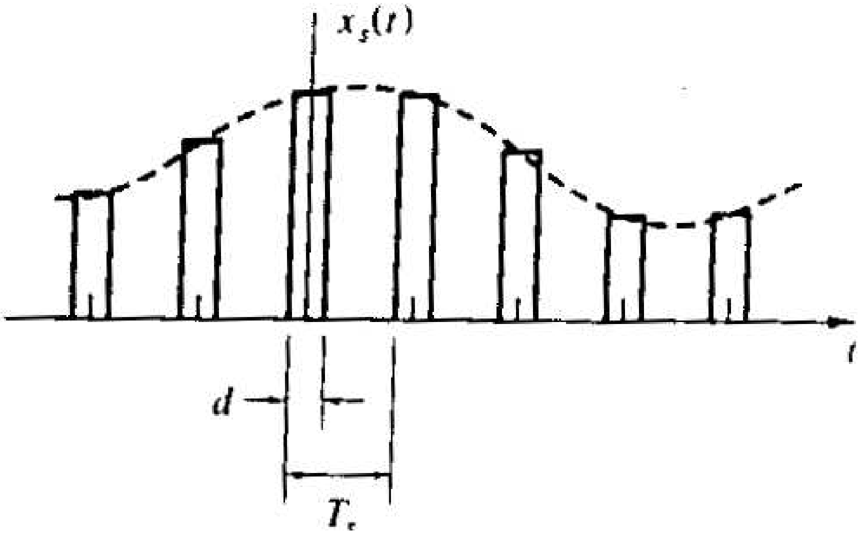

Figure 3: Pulse amplitude modulation

Nyquist's theorem can be shown schematically as shown in fig. 2 The sampling proceedure is mathematically modeled by multiplying the function to be sampled by the comb or sampling function

$$s(t, T_S) = \sum_{n=-\infty}^{n=\infty} \delta(t - nT_S). \tag{3}$$

as shown on the first three pictures in the left (spacial or time) column. This multiplication "loses" all values of the original function that are intermediate between the sampled. The Fourier transform of the sampling function is given, without proof, as

$$s(t, T_S) \Leftrightarrow \frac{1}{T_S} s(f, \frac{1}{T_S}). \tag{4}$$

Thus the multiplication by the sampling function, $m(t)$ by $s(t, T_S)$ in the time domain is a convolution of $M(f)$with $s(f,\, f_s)$,where

$$f_s = \frac{1}{T_S}. \tag{5}$$

This convolution duplicates the frequency spectrum, $M(f)$, at intervals of f_s, as shown in the right column rows 3 and 4. In row 3 we see the case of under-sampling. The spectra overlap so the original function can not be recovered without loss of information. This overlap causes **aliasing** of components of the spectrum. Low frequency components of the spectrum appear as high frequency components and vice versa. In line 4 we see a signal that has been sampled at a sufficient sampling rate, $d < 1/f_m$. Here the band-limited spectra are separated enough to allow loss-less reconstruction of the original signal.

To reconstruct the original signal we have to "filter" out the higher order copies of the spectrum and leave only the zero centered version. This can be done by multiplication in frequency by a box function that will "zero" all but one copy of the original spectrum (line 5 and 6). Of course this operation in the time domain is a convolution of the sampled function with the inverse Fourier transform of the box function which is the since function. This give us the Nyquist reconstruction formula eq. 32.

This instantaneous sampling is not practical, since any sampling takes a finite time. It can be shown that Nyquist's theorem also applies with some small changes for actual finite sampling.

This sampled signal can be used to generate a **pulse amplitude modulated** (PAM) signal, fig. . The sampled signal

$$m_s(t) = m(t)s(t, T_S), \tag{6}$$

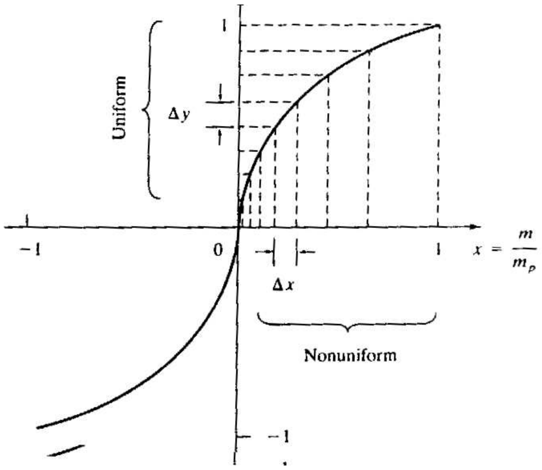

Figure 4: μ-**law** quantization

convoluted with the rectangular pulse

$$p(t) = \begin{cases} 1 & |t| < d/2 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

gives us the pulse modulated signal.

3 Quantizing

For systems that are binary-digital based, quantization is normally based on a range of 2^n different possible values per sample. The simplest quantizing process is to first translate the value to be strictly positive. We then multiply the signal such that it's maximum value is less than 2^n. To quantize the value we simply add .5 and round down to the nearest integer. Each quantizing interval spans Δ values where

$$\Delta = \frac{R}{2^n}. \tag{8}$$

and R is the original range, $m_{max} - m_{min}$ of the signal.

The quantizing introduces a **quantizing error** or **quantizing noise** which can easily be computed assuming that the errors random within the quantizing interval

$$\langle q_e^2 \rangle = \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} q_e^2 dq_e = \frac{\Delta^2}{12} = \frac{R^2}{3 \cdot 2^{n+1}} \tag{9}$$

For voice signal this quantization is not optimal since statistically smaller amplitudes in the signal predominate. Thus the dynamic range of the signal is compressed to give smaller amplitudes greater representation. The particular form of a compression law that has been standardized in North America is the μ**-law,** shown in fig. 4, defined by

$$y = \frac{\ln(1 + \mu |x/x_{max}|)}{\ln(1 + \mu)} \mathrm{sgn}(x) \tag{10}$$

where x is the signal value and $x_{max}$is the maximum value of x. The function sgn is just the sign function

$$sgn(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}. \tag{11}$$